Two-Connected Graphs with Given Diameter

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Abstract: The problem we study in this work is an extremal problem arising from graph theory : what is the minimum number of edges of a 2-connected graph satisfying diameter conditions. This problem deals with survivable netowrk design when the network is subjected to satisfy grade of service constraints. One way to provide networks working when some failures arise is to provide a sufficient connectivity to the networks. Due to equipment robustness 2-connectivity or 2-edge-connectivity will be sufficient. Notice that k-connected networks, k$\geq$3, will provide too expensive networks. Another important parameter is the crossing-delay, ie the total amount of time spent in the network by some data packet to reach its destination from its origin. In order to keep this crossing-delay under reasonable values one can bound the number of hops of a routing path. This leads to bound the diameter of the underlying graph. We prove the following bounds : if G is 2-(vertex)-connected, then $|E| \geq \lceil \frac{nD-(2D+1)}{D-1} \rceil$, if G is 2-edge-connected of odd diameter, then $|E| \geq \lceil \frac{nD-(2D+1)}{D-1} \rceil$, if G is 2-edge-connected of even diameter, then $|E| \geq min(\lceil \frac{nD-(2D+1)}{D-1} \rceil, \lceil \frac{(n-1)(D+1)}{D} \rceil)$.

Key-words: connectivity, diameter, minimum cost

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Graphes 2-connexes de diamètre fixé

Résumé : Dans ce travail, nous étudions un problème extremal de théorie des graphes : quel est le coût minimal en arêtes d'un graphe 2-connexe vérifiant une condition sur le diamètre. Ce problème concerne le design de réseaux tolérants aux pannes lorsque le réseau doit fourinir une certaine qualité de service. Une manière d'avoir un réseau fonctionnant lorsque des pannes surviennent est de garantir une certaine connectivité au réseau. La résistance des équipements induit qu'une 2-connexité ou 2-arête -connexité est suffisante, de plus réseaux k-connexes, avec k$\geq$ 3 coûtent trop cher. Un autre paramètre crucuial est le délai de transmission, c'est à dire le temps total mis par un paquet pour arriver à sa destination depuis son origine. Pour garder ce délai dans une fourchette raisonnable, on peut borner le nombre de sauts nécessaires pour un chemin. Cela revient à borner le diamètre du graphe sous-jacent. Nous prouvons les bornes suivantes : si G est 2-(sommet)-connexe, alors $|E| \geq \lceil \frac{nD-(2D+1)}{D-1} \rceil$, si G est 2-arête-connexe de diamètre impair, alors $|E| \geq \lceil \frac{nD-(2D+1)}{D-1} \rceil$, si G est 2-arête-connexe de diamètre pair, alors $|E| \geq min(\lceil \frac{nD-(2D+1)}{D-1} \rceil, \lceil \frac{(n-1)(D+1)}{D} \rceil)$.

Mots-clés : connexité, diamètre, coût minimal

1 Introduction

The problem we study in this article is an extremal problem arising from graph theory : what is the minimum number of edges of a 2-connected graph satisfying diameter conditions. This problem deals with survivable netowrk design when the network is subjected to satisfy grade of service constraints.

1.1 Motivations

One way to provide networks working when some failures arise is to provide a sufficient connectivity to the networks. Due to equipment robustness 2-connectivity or 2-edge-connectivity will be sufficient. Notice that k-connected networks, k$\geq$3, will provide too expensive networks. Another important parameter is the crossing-delay, ie the total amount of time spent in the network by some data packet to reach its destination from its origin. In order to keep this crossing-delay under reasonable values one can bound the number of hops of a routing path. This leads to bound the diameter of the underlying graph.

1.2 Formulation

A graph $G = (V, E)$ is said k-connected if $G - S$ remains connected for any subset $S \subset V$ such that $|S| \leq k - 1$. The diameter of a graph is D if for each pair of vertices $\{x, y\}$ there is a path containing at most D edges between them, and if there is at least one pair for which the shortest path contains exactly D edges.

We say that a graph satisfies the (n, k, D) property if its order is n ($|V| = n$), if it is k-connected, and if its diameter is D.

Two cost-reduction problems related to fault-tolerant networks are mainly studied

- find a graph $G = (V, E)$ such that E is minimum and G satisfies the (n, k, D) property :

$$P_1(n, k, D) = \begin{cases} Min|E| \\ G = (V, E) \in \{(n, k, D)graphs\} \end{cases}$$

- find a graph $G = (V, E)$ which satisifies the (n, k, D) property such that $< w.E >$ is minimum (weighted problem) :

$$P_w(n, k, D) = \begin{cases} Min\ w(E) = \Sigma_{e \in E} w_e \\ \forall e \in E, w_e \geq 0 \\ G = (V, E) \in \{(n, k, D)graphs\} \end{cases}$$

In [MV64] (1964) Murty and Vijaya settled the problem of determining the minimum number of edges $f_v(n, D, D', k - 1)$ of a graph of order n, of diameter D, and such that its diameter increases of D' at most after removal of $k - 1$ vertices. This latter problem is the $P_1(n, k, D)$ problem when $D' = n - 1$. Indeed, a solution of the $f_v(n, D, n - 1, k - 1)$ problem remains connected (its diameter is less than n) after removal of $k - 1$ edges.

In [Bol68] (1968) Bollobás conjectured that the graph obtained by joining two vertices with p paths of length D were extremal in the $(n, D, D', 1)$ class, when n and D satisfy $n \equiv 2(mod.D - 1)$ and $D' = 2D - 2$. This conjecture has been proven by Cacetta in [Cac76] (1976) in the case

$(n, 4, D', 1)$, $D' \geq 6$. The problem has then been completely resolved by Murty [Mur68] [Mur69] in the cases $D = 2$, $D' \geq 3$ and $k = 2$. He proves that $f_v(n, 2, D', 1) = 2n$-5.

[BBPP83] mentions a result from Usami : $f_v(n, D, 6, 2) = Max\{n, \lceil \frac{4n-8}{3} \rceil\}$. The problem and the general conjecture were also proposed and studied by Enomoto and Usami [EU83] and by A. Laugier, F. Boyer and O. Goldschmidt [LBG97].

We will prove the following bounds : if G is 2-(vertex)-connected, then $|E| \geq \lceil \frac{nD-(2D+1)}{D-1} \rceil$, if G is 2-edge-connected of odd diameter, then $|E| \geq \lceil \frac{nD-(2D+1)}{D-1} \rceil$, if G is 2-edge-connected of even diameter, then $|E| \geq min(\lceil \frac{nD-(2D+1)}{D-1} \rceil, \lceil \frac{(n-1)(D+1)}{D} \rceil)$.

In section 2, we introduce the concept of ear-partition which will be useful in the remainder of the paper. In section 3, we expose 2-(vertex)-connected graphs which are exactly in the bound. In section 4, we show the bound for 2-(vertex)-connected graphs (extended proofs in Appendix). In section 5, we expose 2-edge-connected graphs which are exactly in the bound. In section 6, we show the bound for 2-edge-connected graphs (extended proof in Appendix).

1.3 Simple observations

If $D = 1$, G is a complete graph ; it contains $\frac{|V|(|V|-1)}{2}$ edges. If G is 2-(vertex)-connected or 2-edge-connected, all its vertices are of degree at least 2. Thus $|E| \geq |V|$. The cycles of size $2D$ and $2D + 1$ are 2-(vertex)-connected graphs with minimum number of edges for both problems.

1.4 Notations

Let $G = (V, E)$ an undirected graph. For all $x \in V$ we call $deg(x)$ the degree of x. For all $\{x, y\} \subset V$ we call $D(x, y)$ the distance between x and y in G. We call D the diameter of G ($D = max_{\{x,y\} \subset V} D(x, y)$). For all $S \subset V$ we call $D(x, S) = min_{y \in S} D(x, y)$ ($D(x, \emptyset) = +\infty$). We call $G - S$ the subgraph $(V \backslash S, E \backslash \{(xy) | x \in S\})$. For all subgraph $\Gamma = (V_\Gamma, E_\Gamma)$ of G, we call $G - \Gamma$ the subgraph $G - V_\Gamma$.

Definition 1 (path)
Let $x_0, ..x_k \in V$ and $e_1, ..e_k \in E$ such that $\forall i \in [1, k]$, $e_i = (x_{i-1}x_i)$. Then $\mu = e_1e_2...e_k$ is called path of length k linking x_0 to x_k. $V_\mu = \{x_1, ..x_{k-1}\}$ and $E_\mu = \{e_1, ..e_k\}$.

Definition 2 (ear)
*Let $x_0, ..x_k \in V$ and $e_1, ..e_k \in E$ such that $\forall i \in [1, k]$, $e_i = (x_{i-1}x_i)$ and $\forall i \in [1, k-1]$, $deg(x_i) = 2$. Then $\mu = e_1e_2...e_k$ is called ear of length k linking x_0 to x_k. $V_\mu = \{x_1, ..x_{k-1}\}$,
$E_\mu = \{e_1, ..e_k\}$ and $G - \mu = (V \backslash V_\mu, E \backslash E_\mu)$.*

Remark : a single edge is an ear of length 1.

Definition 3 (decomposition)
Let $S \subset V$ and let $P'_1 = (V_{P'_1}, E_{P'_1}), P'_2 = (V_{P'_2}, E_{P'_2}), ...$ be the connected components of $G - S$. We set $V_{P_i} = V_{P'_i} \bigcup S$ and $E_{P_i} = \{(xy) \in E | \{x, y\} \subset V_{P_i}\}$. $P_1 = (V_{P_1}, E_{P_1}), P_2 = (V_{P_2}, E_{P_2}), ...$ is called a decomposition of G through S.

Remark : the decomposition is empty if $S = V$.

Definition 4 (merging)
*Let $\{x, y\} \subset V$, $x \neq y$. The graph $\frac{G}{x \sim y}$ obtained by merging x and y, is equal to
$(V - \{y\}, (E_{G-\{y\}} \bigcup \{(xz) \mid z \in (V - \{x, y\})$ and $(yz) \in E\})$).*

2 Ear-partition

In this section, we show that a 2-ege-connected graph G can be partitionned into a given subgraph Γ and ears of bounded length β. We also give a property over β (claim 1).

Definition 5 ($\beta(G,S)$)
Let $G=(V,E)$ be an undirected graph. Let S be a non-empty subset of V and let $M = max_{x\in V} D(x,S)$. If $|S|=1$ or if $\exists\, s\in S$ such that $G-\{s\}$ is not connected, then $\beta(G,S)=2M+1$. Otherwise, if $\exists\, x_1,x_2\in V$ and $s_1\neq s_2\in S$ such $D(x_1,s_1)=M$, $D(x_1,S\backslash\{s_1\})>M$, $D(x_2,s_2)=M$ and $D(x_2,S\backslash\{s_2\})>M$ then $\beta(G,S)=2M+1$. Else $\beta(G,S)=2M$.

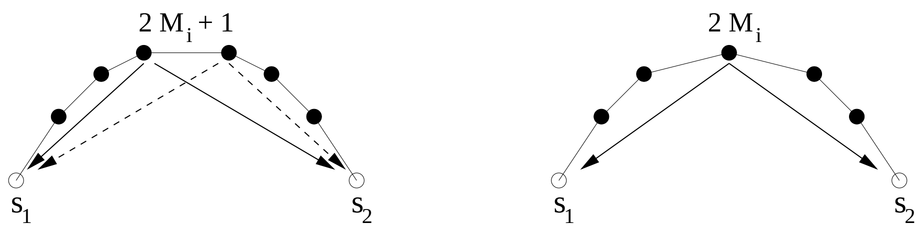

Figure 1: β

Claim 1 (diameter) *Let $G=(V,E)$ be an undirected connected graph and let $S\subset V$ such that $|S|\geq 2$. Let $P_1=(V_{P_1},E_{P_1}), P_2=(V_{P_2},E_{P_2}),\ldots$ be a decomposition of G through S. Let $\beta_i=\beta(P_i,S)$. Then $\forall i\neq j$, $\beta_i+\beta_j\leq 2D+1$*

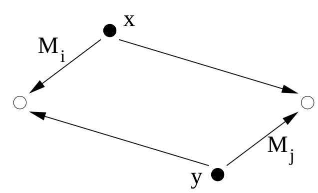

Figure 2: $\beta_i+\beta_j$

Proof in Appendix □

Claim 2 (ear-partition) *Let $G=(V,E)$ be an undirected 2-edge-connected graph. Let $\Gamma=(V_\Gamma,E_\Gamma)$ be a non-null subgraph of G such that $E_\Gamma=\{(xy)\in E|\{x,y\}\subset V_\Gamma\}$ and $\beta=\beta(G,V_\Gamma)$.*

Then G contains $(|V_\Gamma|+\ell)$ vertices and $(|E_\Gamma|+k)$ edges, with $k(\beta-1)\geq \ell\beta$.

Proof in Appendix □

3 The bound is tight for 2-(vertex)-connected graphs

We give a construction for 2-(vertex)-connected of given order and diameter, which have the exact number of edges given by the bound.

Theorem 1 *Let $D\geq 2$ and $n\geq 2D+1$. There is a 2-(vertex)-connected graph $I_{D,n}$ of diameter D which contains exactly n vertices and $\lceil\frac{nD-(2D+1)}{D-1}\rceil$ edges.*

Proof
We call k and m the quotient and the remainder of the euclidian division of $n-(D+2)$ by $(D-1)$ so we have $n = 2+D+k(D-1)+m$. We call $I_{D,n}$ the graph defined as follows : x and y two particular vertices, one ear of length $D+1$ linking x to y, k ears of length D linking x to y, and if $m > 0$, one ear of length $m+1$ linking x to y. $I_{D,n}$ contains n vertices and $\lceil \frac{nD-(2D+1)}{D-1} \rceil$ edges. Observe that $I_{D,n}$ is 2-(vertex)-connected and of diameter D. □

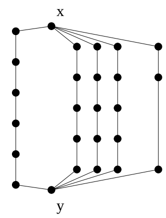

Figure 3: $I_{6,26}$

Note that these are not the only graphs which are in the bound : for instance, there is also the Petersen, and squares : K_4 graphs where each edge has been replaced with an ear of length $2k+1$ ($n = 12k+4$, $D = 4k+1$, $|E| = 12k+6$).

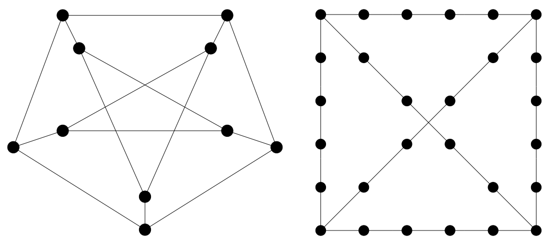

Figure 4: Petersen ; Square with $k = 2$

4 Showing the bound for 2-(vertex)-connected graphs

We will now show by induction that if G is 2-(vertex)-connected, it contains more than $\lceil \frac{|V|D-(2D+1)}{D-1} \rceil$ edges. It is true for the cycles C_k. We assume that every graph containing strictly less than $|E|$ edges does not contradict theorem 2. We call $\Gamma = (V_\Gamma, E_\Gamma)$ a smallest cycle of G. We will study the cases when $G - \Gamma$ is disconnected, when $G - \Gamma$ is connected and $|V_\Gamma| \leq 2D$, and eventually when $|V_\Gamma| \geq 2D+1$.

Lemma 1 (ratio) *Let $\{e, n, D\} \subset \mathbb{N}$ such that $n \geq 3$, if $D \geq 2$, $e \geq \lceil \frac{nD-(2d+1)}{D-1} \rceil$ and if $D = 1$ $e \geq \frac{n(n-1)}{2}$. Let $\{e', n', D'\} \subset \mathbb{N}$ such that $D' \geq D$, $D' \geq 2$, and $(e'-e)(D'-1) \geq (n'-n)D'$. Then $e' \geq \lceil \frac{n'D'-(2d+1)}{D'-1} \rceil$.*

Proof in Appendix □

4.1 Smallest cycle disconnecting G

The following property is a consequence of claim 2.

Claim 3 (disconnect) *Let Γ be a smallest cycle of G. If $G - \Gamma$ is not connected, then* $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$.

Proof in Appendix $\square$

4.2 Smallest cycle of size smaller than $2D$

Lemma 2 (big ear) *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a smallest cycle of G such that $G - \Gamma$ is connected. If an ear of G included in Γ contains strictly more than $\lfloor \frac{|V_\Gamma|-1}{2} \rfloor$ edges, then* $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$.

Proof in Appendix $\square$

4.2.1 Flatening

We introduce here a way to transform a cycle into a tree.

Definition 6 (flatening)
Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a cycle. A flatening of Γ is a pair (T, f) where $T = (V_T, E_T)$ is a tree containing $\lceil \frac{|V_\Gamma|+1}{2} \rceil$ vertices and where f is a composition of $\lfloor \frac{\gamma-1}{2} \rfloor$ mergings such that $f(\Gamma) = T$. The functions f_V and f_E will denote the transformations $f = (f_V : V_\Gamma \to V_T, f_E : E_\Gamma \to E_T)$.

Lemma 3 (legal flatening) *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a cycle and $S \subset V_\Gamma$. If the length of the biggest ear of Γ with no vertex in S is smaller than $\lfloor \frac{V_\Gamma}{2} \rfloor$ then there is a flatening (T, f) of Γ such that for all vertex $s \in V_T$ of degree 1, $s \in f(S)$.*

Proof in Appendix $\square$

4.2.2 Application

We show that a certain flatening on a smallest cycle of G keeps the graph 2-(vertex)-connected. If the cycle is small enough, it boils down to remove an ear of bounded length.

Lemma 4 *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a smallest cycle of G. There is a flatening (T, f) of Γ such that $F(G)$ is a 2-(vertex)-connected graph where F is the extension (as a composition of mergings) of f on G.*

Proof in Appendix $\square$

Claim 4 (size of the cycles) *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a smallest cycle of G. If the size of Γ is smaller or equal to $2D$, then* $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$.

Proof in Appendix $\square$

4.3 Smallest cycle of size greater than $2D + 1$

If the size of every cycle in G is greater than $2D + 1$, we have strong indications on G.

Claim 5 (vertices of degree 3) *If $\exists x \in V$ such that $deg(x) \geq 3$ then $\forall x \in V$, $deg(x) \geq 3$.*

Proof in Appendix $\square$

Claim 5 (vertices of degree 3) gives an indication on the number of edges of G, which is sufficient to conclude when $D \geq 3$. A closer examination concludes also for $D = 2$.

Theorem 2 *Let $G = (V, E)$ be an undirected 2-(vertex)-connected graph of diameter $D \geq 2$. Then $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$.*

Proof in Appendix $\square$

5 The bound is tight for 2-edge-connected graphs

Let $D \in \mathbb{N}^*$ and $n \geq 2D+1$. Notice that $n \geq D^2 + D + 1 \Leftrightarrow \frac{(n-1)(D+1)}{D} \leq \frac{nD-(2D+1)}{D-1}$. If D is odd, or if $n \leq D^2 + D + 1$, theorem 1 implies that $I_{D,n}$ is a 2-edge-connected graph of diameter D, containing exactly n vertices and $\lceil \frac{nD-(2D+1)}{D-1} \rceil$ edges, so the bound is tight. If D is even and if $n \geq D^2 + D + 1$, then we use another construction :

Theorem 3 *Let $D \in 2\mathbb{N}^*$ and $n \geq 1$. There is a 2-edge-connected graph $J_{D,n}$ of diameter D which contains exactly n vertices and $\lceil \frac{(n-1)(D+1)}{D} \rceil$ edges.*

Proof
We call k and m the quotient and the remainder of the euclidian division of $n-1$ by D so we have $n = 1 + kD + m$. We call $J_{D,n}$ the graph defined as follows : x one particular vertex, k ears of length $D+1$ linking x to itself, and if $m > 0$, one ear of length $m+1$ linking x to itself.

$J_{D,n}$ contains n vertices and $\lceil \frac{(n-1)(D+1)}{D} \rceil$ edges. Observe that $J_{D,n}$ is 2-edge-connected and of diameter D. $\square$

6 Showing the bound for 2-edge-connected graphs

A 2-edge-connected graph G is 2-(vertex)-connected, or has a vertex x such that $G - \{x\}$ is not connected. This latter case gives the second term of the bound.

Theorem 4 *Let $G = (V, E)$ be an undirected 2-edge-connected graph (or multigraph) of diameter $D \geq 2$. If D is odd then $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$. If D is even then $|E| \geq min(\lceil \frac{(|V|-1)(D+1)}{D} \rceil, \lceil \frac{|V|D-(2D+1)}{D-1} \rceil)$.*

Proof in Appendix $\square$

7 Further researches

Since we solve the problem for the case $k = 2$ we will look after the minimum number of edges of graphs satifying higher connectivity requirements. One can check that the graphs we propose contain some vertices with very large degree, so in order to provide more robust networks it may be interesting to introduce a bound on the maximum degree of the graph.
We are interested too in finding some other classes of graphs satisfying the bounds shown in this paper. We showed that the bounds are valid for subgraphs of the complete graph, it seems interesting to determine the classes of graphs for which the subgraphs satisfying the $(n, 2, D)$ property verify the bounds. In addition we want to study the cases for which the inequality $\Sigma_{e \in E} x_e \geq \lceil \frac{nD-(2D+1)}{d-1} \rceil$ is facets defining of the polytope of the 2-connected subgraphs having diameter no greater than D.

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Appendix

Here are the proofs which have been removed for sake of brevity of sections Ear-partition (A), Showing the bound for 2-(vertex)-connected graphs (B) and Showing the bound for 2-edge-connected graphs (C).

A Ear-partition

Proof of claim 1 (diameter)
Assume the opposite : let $x \in P_i$, $y \in P_j$ and $\{s_1, s_2\} \subset V_\Gamma$ such that $s_1 \neq s_2$,

$$\begin{array}{ll} D(x, s_1) \geq \lfloor \frac{\beta_i}{2} \rfloor, \quad \forall a \in V_\Gamma \backslash \{s_1\} & D(x, a) \geq \lceil \frac{\beta_i}{2} \rceil \\ D(y, s_2) \geq \lfloor \frac{\beta_j}{2} \rfloor, \quad \forall a \in V_\Gamma \backslash \{s_2\} & D(y, a) \geq \lceil \frac{\beta_j}{2} \rceil. \end{array}$$

We have $D(x, y) \geq min(\lfloor \frac{\beta_i}{2} \rfloor + \lceil \frac{\beta_j}{2} \rceil, \lceil \frac{\beta_i}{2} \rceil + \lfloor \frac{\beta_j}{2} \rfloor) \geq \lfloor \frac{\beta_i + \beta_j}{2} \rfloor \geq \frac{2D+2}{2} \geq D + 1$, which is absurd. □

Lemma 5 *Let $G = (V, E)$ be an undirected graph. Let S be a non-empty subset of V. Then the length of any ear of G which has no vertex in S is smaller or equal to $\beta(G, S)$.*

Proof
We call $e_1 ... e_k$ an ear of G linking x_0 to x_k, where $\forall i \in [1, k-1], x_i \notin S$. We have

$$\begin{array}{ll} D(x_{\lfloor \frac{k}{2} \rfloor}, x_0) \geq \lfloor \frac{k}{2} \rfloor, \quad \forall s \in S \backslash \{x_0\} & D(x_{\lfloor \frac{k}{2} \rfloor}, s) \geq \lceil \frac{k}{2} \rceil \\ D(x_{\lceil \frac{k}{2} \rceil}, x_k) \geq \lfloor \frac{k}{2} \rfloor, \quad \forall s \in S \backslash \{x_k\} & D(x_{\lceil \frac{k}{2} \rceil}, s) \geq \lceil \frac{k}{2} \rceil \end{array}$$

If $\beta(G, S)$ is odd, then $\beta \geq k$. Otherwise, we can not have $x_0 = x_k \in S$ so $\beta \geq k$ as well. □

Proof of claim 2 (ear-partition)
We call $D_k = \{x \in V$ such that $D(x, V_\Gamma) = k\}$. So we have $D_0 = V_\Gamma$, $D_1, ... D_{\lfloor \frac{\beta}{2} \rfloor}$.

- If β is odd, then we will do a partition satisfying the following conditions :
  - $G = (\bigcup_i V_i, \bigcup_i E_i)$; $\forall i \neq j$, $V_i \bigcap V_j = \emptyset$ and $E_i \bigcap E_j = \emptyset$
  - $V_0 = V_\Gamma$ and $E_0 = E_\Gamma$
  - $\forall i \geq 1$, $|V_i| \leq (|E_i| - 1)$ and $|E_i| \leq \beta$

  We add another condition to the partition :

  - $(xy) \in E_i \Rightarrow \{x, y\} \subset \bigcup_{j \leq i} V_j$

  For all $(i \geq 1)$ we set (V_i, E_i) by induction : if $(\bigcup_{j<i} V_j) = V$ and $(\bigcup_{j<i} E_j) \neq E$, we choose an edge e in $E \backslash (\bigcup_{j<i} E_j)$ and we set $(V_i, E_i) = (\emptyset, \{e\})$.
  Otherwise $(\bigcup_{j<i} V_j) \neq V$:

  - let $k_{max} = max(k \mid D_k \backslash (\bigcup_{j<i} V_j) \neq \emptyset)$
  - let $x \in (D_{k_{max}} \backslash \bigcup_{j<i} V_j)$
  - we choose a shortest path μ_1 from x to $(\bigcup_{j<i} V_j)$. Since $V_\Gamma \subset (\bigcup_{j<i} V_j)$, the length of this path is smaller than $\lfloor \frac{\beta}{2} \rfloor$.

– since $deg(x) \geq 2$, we choose a vertex y adjacent to x such that (xy) is not on the path μ_1. If $y \in (\bigcup_{j<i} V_j)$, we set $V_i = (V_{\mu_1} \bigcup \{x\})$ and $E_i = (E_{\mu_1} \bigcup \{(xy)\})$. If $y \notin (\bigcup_{j<i} V_j)$, $y \in D_k$ where $k \leq k_{max}$: x is not on the shortest path from y to V_Γ. So there is a path μ_2 from y to $(\bigcup_{j<i} V_j)$ which does not contain x, and which length is smaller than $\lfloor \frac{\beta}{2} \rfloor$. We set $V_i = (V_{\mu_1} \bigcup V_{\mu_2} \bigcup \{x, y\})$ and $E_i = (E_{\mu_1} \bigcup E_{\mu_2} \bigcup \{(xy)\})$.

- If β is even and if $\forall (xy) \in E$, $\{x, y\} \not\subset D_{\frac{\beta}{2}}$ then we do the partition as if β was odd.
- If β is even and if $\exists (xy) \in E$ such that $\{x, y\} \subset D_{\frac{\beta}{2}}$ then we call μ the biggest ear with no vertex in V_Γ containing (xy). If $\forall\ s \in V_\Gamma$, $G - \mu - \{s\}$ is connected, then $\beta(G - \mu, V_\Gamma) \leq \beta$ and the length of μ is smaller than β (lemma 5), so the proof is correct by recursion. Otherwise, we call $S' = \{s \in S \mid G - \mu - \{s\}$ is not connected$\}$. We call $X_1 ... X_\ell$ the connected components of $G - \mu - S'$, where μ is linking X_1 to X_ℓ. Since β is even, $(X_1 \bigcup X_\ell) \bigcap S \neq \emptyset$, for instance $X_1 \bigcap S \neq \emptyset$. Let $s \in X_1 \bigcap S$, $x_0 \in X_1$, $x_1 \in (V_\mu \bigcup X_\ell)$ such that $(x_0 x_1) \in E_\mu$. Consider $G' = (V, E \backslash \{(x_0 x_1\} \bigcup \{(s x_1)\})$. G' is 2-edge-connected, $\beta(G', V_\Gamma) \leq \beta$ so the proof is correct by recursion.

By counting the vertices and the edges through this partition we conclude the proof. □

B Showing the bound for 2-(vertex)-connected graphs

Proof of lemma 1 (ratio)

If $D = 1$, we have $e \geq \lceil \frac{2n-5}{1} \rceil$ (as if D equalled 2); if $D \geq 2$, we have $e \geq \lceil \frac{nD-(2D+1)}{D-1} \rceil$. Since $n \geq 3$ this implies

$$\begin{array}{rcl} e & \geq & \lceil \frac{nD'-(2D'+1)}{D'-1} \rceil \\ e' & \geq & \lceil \frac{nD'+(e'-e)(D'-1)-(2D'+1)}{D'-1} \rceil \end{array}$$

Since $(e' - e)(D' - 1) \geq (n' - n)D'$, we have

$$e' \quad \geq \quad \lceil \frac{n'D'-(2D'+1)}{D'-1} \rceil$$

□

B.1 Smallest cycle disconnecting G

Lemma 6 *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a smallest cycle in G. Let $P_1 = (V_{P_1}, E_{P_1}), P_2 = (V_{P_2}, E_{P_2}), ...$ be a decomposition of G through V_Γ. Then P_i is 2-(vertex)-connected and its diameter is smaller or equal to D.*

Proof

Let $x \in V_{P_i}$. If $x \in V_\Gamma$, $(P_i - \Gamma)$ is connected to Γ by at least two vertices, so $P_i - \{x\}$ remains connected. If $x \notin V_\Gamma$, each connected component of $(P_i - \Gamma) - \{x\}$ is connected to Γ, so $P_i - \{x\}$ is connected. This proves that P_i is 2-(vertex)-connected.

Let $y \in V_{P_i} \backslash \{x\}$. If a shortest path from x to y is included in P_i, then the distance between x and y is the same in P_i and in G. Otherwise, there is a shortest path $\mu_1 \mu_2 \mu_3$ from x to y with μ_1, μ_3 included in P_i and μ_2 with no edge in P_i linking $a \in V_\Gamma$ to $b \in V_\Gamma$. μ_2 is a shortest path from a to b strictly shorter than any path included in Γ. This contradicts the fact that Γ is a cycle of minimum size. This proves that the diameter of P_i is smaller or equal to D. □

Proof of claim 3 (disconnect)
Let $P_1 = (V_{P_1}, E_{P_1}), P_2 = (V_{P_2}, E_{P_2}), ...$ be a decomposition of G through V_Γ. We call $\beta_i = \beta(P_i, V_\Gamma)$, $\ell_i = |V_{P_i}| - |V_\Gamma|$ and $k_i = |E_{P_i}| - |E_\Gamma|$. Claim 1 (diameter) shows that one P_i at most (say P_1) satisfies $\beta_i \geq D + 1$. P_1 contains $|V| - \ell$ vertices and $|E| - k$ edges, where $\ell = \Sigma_{i \neq 1} \ell_i$ and $k = \Sigma_{i \neq 1} k_i$. According to claim 2 (ear-partition), $\forall i \neq 1$ $k_i(\beta_i - 1) \geq \ell_i \beta_i$, which implies $k(D-1) \geq \ell D$. According to lemma 6, P_1 is 2-(vertex)-connected and its diameter is smaller or equal to D, so we have by induction $|E_{P_1}| \geq \lceil \frac{|V_{P_1}|D-(2D+1)}{D-1} \rceil$. Lemma 1 (ratio) implies that $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$. $\square$

B.2 Smallest cycle of size smaller than $2D$

Lemma 7 *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a smallest cycle in G. If the length of any ear of G included in Γ is smaller or equal to D, then $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$.*

Proof
Assume that $e_0 e_1 ... e_D$ is an ear of length $D+1$ included in Γ, linking x_0 to x_{D+1} :

$$\begin{array}{ll} \forall x \in V, & D(x, x_{\lfloor \frac{D+1}{2} \rfloor}) \leq D) \text{ and } D(x, x_{\lceil \frac{D+1}{2} \rceil}) \leq D) \\ \forall x \in V \backslash \{x_1, x_2 .. x_D\}, & min(D(x, x_0), D(x, x_{D+1})) \leq \lceil \frac{D-1}{2} \rceil \end{array}$$

This implies that $\beta(G, V_\Gamma) \leq D$. According to claim 2 (ear-partition), $|V_\Gamma| = |V| - \ell$ and $|E_\Gamma| = |E| - k$ with $k(D-1) \leq \ell D$. According to lemma 1(ratio), $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$. $\square$

Lemma 8 *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a smallest cycle in G such that $G - \Gamma$ is connected. Let μ be a longest ear of G included in Γ. Then $G - \mu$ is connected.*

Proof
Let $s_1, s_2 \in V_\Gamma$ be two vertices of degree greater or equal to 3. Let μ be an ear of G included in Γ linking s_1 to s_2. Let $x \in V \backslash V_\mu$. If $x \in V_\Gamma$, $(\Gamma - \mu) - \{x\}$ has at most two connected components, each of them is connected to $G - \Gamma$. Since $G - \Gamma$ is connected, $(G - \mu) - \{x\}$ is connected as well. If $x \notin V_\Gamma$, since G is 2-(vertex)-connected, each connected component of $(G - \Gamma) - \{x\}$ is connected to $\Gamma - \mu$. Therefore $(G - \mu) - \{x\}$ is connected. $\square$

Proof of lemma 2 (big ear)
Let μ be a longest ear of G included in Γ. If its length is greater than $D+1$ then lemma 7 tells that $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$. Otherwise, according to lemma 8, $G - \mu$ is a 2-(vertex)-connected graph. It contains $(|V| - \ell)$ vertices and $(|E| - k)$ edges, with $k(D-1) \geq \ell D$. If the length of μ is greater than $\lceil \frac{|V_\Gamma|}{2} \rceil$, then the diameter of $G - \Gamma$ is smaller or equal to D, so according to lemma 1 (ratio), $|E| \geq \lceil \frac{|V|D-(2d+1)}{D-1} \rceil$. $\square$

B.2.1 Flatening

Proof of lemma 3 (legal flatening)
If $|V_\Gamma|$ is even then let s_1 and s_2 be vertices of S such that $D(s_1, s_2)$ is maximal.

- If $D(s_1, s_2) = \frac{|V_\Gamma|}{2}$, we name the vertices of T this way : $V_T = \{t_0, t_1 ... t_{\frac{\gamma}{2}}\}$, and we set

$$f_V : x \to t_{D(x, s_1)}.$$

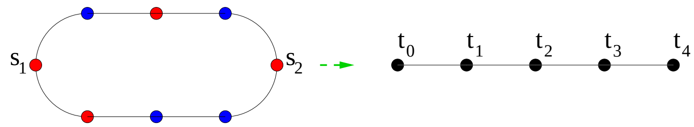

Figure 5: Flatening 1

- Otherwise S contains a vertex s_3 such that each connected component of $\Gamma - \{s_1, s_2, s_3\}$ contains at most $(\frac{|V_\Gamma|}{2} - 2)$ vertices. We define $D_1 = \frac{D(s_1,s_2)+D(s_1,s_3)-D(s_2,s_3)}{2}$, $D_2 = \frac{D(s_1,s_2)+D(s_2,s_3)-D(s_1,s_3)}{2}$ anD $D_3 = \frac{D(s_1,s_3)+D(s_2,s_3)-D(s_1,s_2)}{2}$. We have $D_1 + D_2 + D_3 = \frac{|V_\Gamma|}{2}$. We set $V_T = \{t, t_1^0..t_1^{D_1-1}, t_2^0..t_2^{D_2-1}, t_3^0..t_3^{D_3-}$ anD $f_V : x \rightarrow \begin{cases} t_i^j \text{ if } D(x, s_i) = j < D_i \\ t \text{ if } D(x, s_1) = D_1,\ D(x, s_2) = D_2 \text{ or } D(x, s_3) = D_3 \end{cases}$

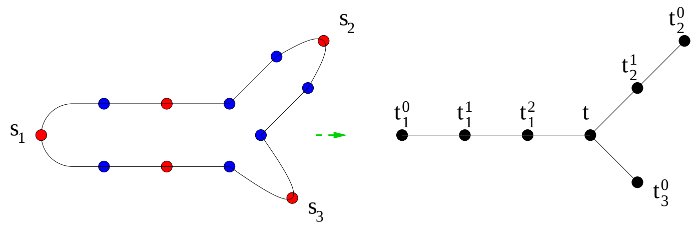

Figure 6: Flatening 2

If $|V_\Gamma|$ is odd we merge (merging called f_1) two consecutive vertices of Γ : we obtain a cycle Γ' of even size $(|V_\Gamma| - 1)$. The length of any ear of Γ' with no vertex in $f_1(S)$ is at most $\frac{|V_\Gamma|-1}{2}$ so there is a flatening (T, f) of Γ' such that for all x in Γ of degree 1, $f^{-1}(x) \bigcap f_1(S) \neq \emptyset$. Thus $(T, f \circ f_1)$ is a flatening of Γ related such that for all x in Γ of degree 1, $(f \circ f_1)^{-1}(x) \bigcap S \neq \emptyset$. $\square$

B.2.2 Application

Proof of lemma 4

Let $S = \{x \in V_\Gamma \mid deg(x) \geq 3\}$. According to lemma 2 (big ear), the length of the biggest ear of Γ with no vertex in S is smaller than $\lfloor \frac{|V_\Gamma|}{2} \rfloor$. Therefore, we have a flatening (claim 3) $(T = (V_T, E_T), f)$ of Γ related to S. We naturally extend (as a composition of mergings) f in $F = (F_V, F_E)$ on G. Let $x \in F_V(V)$. If $x \notin V_T$, then $F_V^{-1}(x)$ contains a single vertex : $G - F_V^{-1}(x)$ is connected, and so is $F(G) - \{x\}$. If $x \in V_T$, then $T - \{x\}$ is a reunion of trees. Each tree is connected to $F(G - \Gamma)$. Since $G - \Gamma$ is connected, so is $F(G) - \{x\}$. $\square$

Proof of claim 4 (size of the cycles)

Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a smallest cycle of G. According to lemma 4, there is a flatening (T, f) of Γ such that $F(G)$ is a 2-(vertex)-connected graph where F is the extension (as a composition of mergings) of f on G. Since F is a composition of mergings, the diameter of $F(G)$ is smaller or equal to D. Moreover, $F(G)$ contains $(|V| - \ell)$ vertices and $(|E| - k)$ edges, where $\ell = \lfloor \frac{|V_\Gamma|-1}{2} \rfloor$ and $k \geq \ell + 1$. According to lemma 1 (ratio), we must have $k(D-1) < \ell D$. This implies that $\ell \geq D$, what is to say $|V_\Gamma| \geq (2D+1)$. $\square$

B.3 Smallest cycle of size greater than $2D + 1$

Lemma 9 *If the smallest cycle of G is of size greater than $2D+1$, then each vertex of $V_{\bar{G}}$ belongs to a cycle of size $(2D+1)$.*

Proof
Let $x \in V$ and y one of the most distant vertex from x in $\bar{G}$. We choose a shortest path μ_1 from x to y. Since $deg(y) \geq 2$, we choose a vertex z adjacent to y which is not on μ_1. We choose a shortest path μ_2 from z to x. The path $\mu = \mu_1(yz)\mu_2$ is a path of length $2D+1$ linking x to itself. Since (yz) appears only once in this path, μ contains a cycle which size is smaller than $2D+1$: the size of this cycle is exactly $2D+1$ and x belongs to it. $\square$

Lemma 10 *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a smallest cycle of $\bar{G}$ of size $2D+1$. If G is not a cycle, then $\forall x \in V_\Gamma, deg(x) \geq 3$.*

Proof
Let $a_0, a_2 ... a_{2D}$ be the vertices of Γ such that $E_\Gamma = \{(a_i a_{i+1}), i \in \frac{\mathbb{Z}}{(2D+1)\mathbb{Z}}\}$. Since $G \neq \Gamma$ we assume that $deg(a_D) \geq 3$. Γ is a smallest cycle of G, so $D(a_0, a_D) = D$. Let $x \in V \backslash V_\Gamma$ be a neighbour of a_D and let μ be a shortest path from x to a_0. Since the length of μ is smaller than D, E_μ does not contain $(a_D x)$. Since there is no cycle of size smaller than $2D$, $V_\Gamma \bigcup V_\mu = \emptyset$. Therefore, $deg(a_0) \geq 3$. By symmetry, $deg(a_{2D}) \geq 3$. By symmetry every vertex in the cycle satisfies $deg(a_i) \geq 3$. $\square$

Proof of claim 5 (vertices of degree 3)
According to the two former lemmas, each vertex in G belongs to a cycle of size $(2D+1)$, and if G is not a cycle, each vertex is of degree greater or equal to 3. $\square$

Proof of theorem 2
Le Γ be a smallest cycle of G. If $G = \Gamma$, or if the size of Γ is smaller or equal to $2D$, then according to claims 3 (disconnect) and 4 (size of the cycles) , $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$. Otherwise, according to claim 5 (vertices of degree 3), $\forall x \in V_{\bar{G}}, deg(x) \geq 3$.

- If $D \geq 3$, we have $2|E| \geq 3|V|$, so $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$.
- If $D = 2$, We choose a vertex s of highest degree in G,and we divide the other vertices into two sets : $D_1 = \{x \in G \mid D(x,s) = 1\}$ and $D_2 = \{y \in G \mid D(y,s) = 2\}$ G does not contain any cycle of size smaller or equal to 4. This implies

  $$\begin{array}{ll} \forall \{x_1, x_2\} \subset D_1, x_1 \neq x_2, & (x_1 x_2) \notin E \\ \forall y \in D_2 \ \exists! \ x \in D_1, & (xy) \in E \end{array}$$

  So we divide D_2 into the disjoint union :

  $$D_2 = \bigcup_{x \in D_1} D_2^x \quad \text{with} \quad D_2^x = \{y \in D_2 \mid D(x,y) = 1\}$$

  If $deg(s) = 3$, G contains exactly 10 vertices and 15 edges (G is the Petersen), so $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$. If $deg(s) \geq 4$, we define : $\forall x \in D_1, r_x = deg(x) - 3$ and $r = \Sigma_{x \in D_1} r_x$. We have $|V| = 1 + deg(s) + (2deg(s) + r)$ which is $|V| - 1 = 3deg(s) + r$. Since

  $$\forall x_1 \in D_1, \forall y_1 \in D_2^{x_1}, \forall x_2 \in D_1 \backslash \{x_1\}, \quad D(y_1, x_2) \leq 2$$

  the degree of each vertex in D_2 is greater or equal to $|D_1| \geq 4$. We add the degrees of all the vertices in G :

  $$\begin{array}{rcccccc} 2|E_{\bar{G}}| & = & deg(s) & + & \Sigma_{x \in D_1} deg(x) & + & \Sigma_{y \in D_2} deg(y) \\ 2|E_{\bar{G}}| & \geq & deg(s) & + & 3deg(s) & + & 4(2deg(s) + r) \\ 2|E_{\bar{G}}| & \geq & 4(3deg(s) + r) & & & & \\ 2|E_{\bar{G}}| & \geq & 4(|V| - 1) & & & & \\ |E_{\bar{G}}| & \geq & \lceil \frac{|V|D-(2D+1)}{D-1} \rceil & & & & \end{array}$$

$\square$

C Showing the bound for 2-edge-connected graphs

Proof of theorem 4

If G is 2-(vertex)-connected, according to theorem 2, $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$. Otherwise, there is a vertex s such that $G - \{s\}$ is not connected. Let $P_1 = (V_{P_1}, E_{P_1}), P_2 = (V_{P_2}, E_{P_2}), \ldots$ be a decomposition of G through $\{s\}$, $\beta_i = \beta(P_i, \{s\})$, $\ell_i = |V_{P_i}| - 1$ and $k_i = |E_{P_i}|$. We set P_1 such that β_1 is maximal.

- if $\lfloor \frac{\beta_1}{2} \rfloor < \frac{D}{2}$, according to claim 2 (ear-partition), $\forall i\ k_i(\beta_i - 1) \geq \ell_i \beta_i$, which implies $|E|(D - 1) \geq (|V| - 1)D$ so $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$.
- if $\lfloor \frac{\beta_1}{2} \rfloor = \frac{D}{2}$ (D must be even), then $\forall i\ \beta_i \leq (D+1)$ and according to claim 2 (ear-partition), $\forall i\ k_i(\beta_i - 1) \geq \ell_i \beta_i$, which implies $|E|(D) \geq (|V| - 1)(D+1)$ so $|E| \geq \lceil \frac{(|V|-1)(D+1)}{D} \rceil$.
- if $\lfloor \frac{\beta_1}{2} \rfloor > \frac{D}{2}$, then $\forall i \neq 1$, $\beta_i \leq D$. P_1 contains $|V| - \ell$ vertices and $|E| - k$ edges, where $\ell = \Sigma_{i \neq 1} \ell_i$ and $k = \Sigma_{i \neq 1} k_i$. According to claim 2 (ear-partition), $\forall i \neq 1\ k_i(\beta_i - 1) \geq \ell_i \beta_i$, which implies $k(D-1) \geq \ell D$. Observe that P_1 is 2-(edge)-connected, that its diameter is smaller or equal to D, and that $|E_{P_1}| < |E|$. By recursion, we suppose that theorem 4 holds true for P_1 :
  - if D is odd, then $|E_{P_1}| \geq \lceil \frac{|V_{P_1}|D-(2D+1)}{D-1} \rceil$ so $|E| \geq \lceil \frac{|V|D-(2D+1)}{D-1} \rceil$.
  - if D is even, then $|E_{P_1}| \geq min(\lceil \frac{|V_{P_1}|D-(2D+1)}{D-1} \rceil, \lceil \frac{(|V_{P_1}|-1)(D+1)}{D} \rceil)$, so $|E| \geq min(\lceil \frac{|V|D-(2D+1)}{D-1} \rceil, \lceil \frac{(|V|-1)(D+1)}{D} \rceil)$.

$\square$

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